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Duality from topological symmetry

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Abstract: We describe topological gauge theories for which duality properties are encoded by construction. We study them for compact manifolds of dimensions four, eight and two. The fields and their duals are treated symmetrically, within the context of field–antifield unification. Dual formulations correspond to different gauge-fixings of the topological symmetry. We also describe novel features in eight-dimensional theories, and speculate about their possible "Abelian" descriptions.

Keywords: Duality in Gauge Field Theories, Superstrings and Heterotic Strings, Topological Field Theorie.

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1. Introduction

The relationship between topological gauge theories for four-manifold invariants and physical supersymmetric theories has attracted a lot of interest. A breakthrough has occurred through the work of Seiberg and Witten on an exact low-energy description of the physical $D = 4, N = 2$ supersymmetric Yang–Mills theory [1]. This theory can be topologically twisted in the weak coupling ultraviolet regime to produce the integral over instanton moduli space of certain characteristic classes for the correlation function of topological observables. This provides a description of four-manifold invariants, although a precise compactification procedure of the moduli space instantons is not known in each case. By going to the infrared description of the same theory, which is an Abelian gauge theory with exact electromagnetic duality, integrals over the instanton moduli space are found to equate integrals over the moduli space of vacua [2, 3, 4, 5]. The generating function of topological correlators can be written in terms of Seiberg–Witten monopole invariants and of the deformed holomorphic prepotential $\mathcal{F}(a, t)$. The function $\mathcal{F}(a, 0)$ has a description in terms of the (holomorphic) symplectic geometry on C^{2r}, where r is the rank of a gauge group. Actually, $\mathcal{F}(a, 0)$ is a generating function of Γ-invariant submanifold in C^{2r}, where Γ is the finite modular subgroup (electromagnetic duality group) of $\mathrm{SP}(2r, \mathbb{Z})$. This

submanifold is described by a function $\mathcal{F}$ in the following way. Denote by ω the holomorphic symplectic form on C^{2r} and take $\theta = d^{-1}\omega$. Then the restriction of θ on the lagrangean submanifold is defined by $\mathcal{F}$ such that $\theta|_{\mathcal{L}} = d\mathcal{F}$. The β-function of the $N=2$ gauge theory provides the asymptotic behavior of the prepotential, and the positivity of Imτ is required, with $\tau = \partial^2\mathcal{F}$. The prepotential $\mathcal{F}(a,t)$, which describes the correlators of observables from $H^*(X)$, is defined as a solution of some Hamiltonian evolution of $\mathcal{F}(a,0)$, which is then a deformation of the lagrangean submanifold.

In this paper we will demonstrate that the language of TQFTs [6, 7] is adequate to describe the properties related to the duality symmetry, in a way that extends refs. [8, 9]. It provides the symmetric treatment of fields and their duals, and we will see how different formulations, and/or dual pictures, can appear in the process of different gauge-fixings of the enlarged topological symmetry that act on all fields. The doubling of fields, which allows various formulations, can be generalized to other theories, in dimensions other than four, and for gauge fields that can be forms of degrees different than 1. The naturalness of this doubling is generally dictated by the principle of kinematical ghost and gauge field unification. We will actually study topological theories in two and eight dimensions, for which the approach developed for the four-dimensional case turns out to be generalizable and useful. Moreover, dimensional reduction can be applied to our results, the most interesting case being, in our opinion, the Yang–Mills TQFT in eight dimensions.

For eight-manifolds with $Spin(7)$ holonomy, the moduli space of four-dimensional instantons is replaced by that of octonionic instantons, which, unfortunately, is yet widely unexplored. The eight-dimensional TQFT that was constructed in [7, 10] is, however, quite an attractive theory. In particular, its dependence on an invariant self-dual four-form Ω_4 makes it very interesting, since, from a physical point of view, such four-forms could be related to propagating four-forms in ten dimensions. The untwisted version of this TQFT is nothing else than the ordinary eight-dimensional Yang–Mills supersymmetric theory, i.e. the dimensional reduction to eight dimensions of the $N = 1, D = 10$ super-Yang–Mills theory [7]. Such a theory also determines the matrix string in the light-cone gauge [11, 12], after dimensional reduction to two dimensions, and a certain form of the Seiberg–Witten theory in four dimensions.

Our treatment will clarify already known observations for the four-dimensional case. In the eight-dimensional case, it will give interesting new features. The eight-dimensional Yang–Mills theory is not renormalizable as such. Thus, some of the arguments used in four dimensions cannot be applied directly. Moreover, the genuine physical Yang–Mills theory on R^8 is infrared-free, and it might be concluded that non-trivial statements using infrared description cannot be made.[1]

[1]Notice that the eight-dimensional Yang–Mills theory is not fully topological, since it is only independent of the metric variations which do not change the $Spin(7)$ structure.

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We will speculate on possible resolutions of these questions. First, we will show the possibility of a dual formulation in eight dimensions. It relies on the existence of a $Spin(7)$-invariant four-form Ω_4, which is already known to permit the construction of a Yang–Mills TQFT in eight dimensions [7]. (The existence of Ω_4 amounts to that of octonionic structure coefficients in a local description.) Here, we will show that, thanks to Ω_4, duality can be established between a pair of one-forms in eight dimension, and one finally obtains a duality in eight dimensions, which is parallel to that of four dimensions. The argument will be that the dual in eight dimensions of a Yang–Mills field A, which is a five-form gauge field A_5, can be restricted to a dual one-form A_D, by a partial gauge-fixing of the topological gauge symmetry, which amounts to "divide" A_5 by Ω_4, $A_5 = \Omega_4 A_D$, as well as some of the ghosts.

Secondly, we will see that, by handling the duality of supersymmetric theories in the context of TQFTs, we can accommodate the existence of higher-order interactions in the Yang–Mills curvatures.

A natural way of thinking about such higher-order corrections, which add to the lagrangean operators like quartic powers of the Yang–Mills curvature, is the string theory compactification down to eight dimensions. The latter, permits the replacement of the ultraviolet cut-off of the eight-dimensional theory by the string tension parameter α'. This certainly modifies the infrared properties and gives a physical content to the eight-dimensional TQFT.

A novel feature of our work is that we will introduce the $\mathcal{O}(\alpha')$ quartic interactions (and possibly the higher-order ones, if we were to consider higher-order α' corrections), by using $dA + \mathcal{O}(\alpha') t_8 dAdAdA$ instead of the genuine Yang–Mills two-form curvature dA in the octonionic gauge function of the Yang–Mills TQFT. Here, t_8 is an invariant SO(8) tensor with eight indices, which is is proportional to the trace of a product of γ matrices.

In the way we will proceed, the complete string-corrected eight-dimensional lagrangean is still an s-exact term, with a topological gauge-function that is not in contradiction with the notion of "holomorphicity". Requirement of supersymmetry and explicit one-loop-order computations provide the explicit form of t_8 [13, 14]. We suggest that t_8 is a functional of Ω_4. We find it appealing to believe that Ω_4 is a reminder of a propagating four-form gauge field in ten dimensions, which has been gauge-fixed equal to a background self-dual four-form in eight dimensions, using a ten-dimensional topological symmetry.

Another way of seeing the relevance of higher-order corrections is to compactify the eight-dimensional theory down to a four-dimensional compact manifold, say $K3$. If we integrate out all massive modes, this gives a (non-local) four-dimensional gauge theory for which the infrared description is given by an Abelian gauge theory with some duality group Γ. The possibility of defining the eight-dimensional theory on the product space $X \times K3$ in terms of the four-dimensional renormalization group might lead us to the conclusion that there should be an analogous Abelian description

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in eight dimensions. Quartic terms contribute to the four-dimensional prepotential upon compactification, and we conclude that the truly eight-dimensional features of the theory cannot be captured unless these terms are included in the discussion (plus corrections to all orders in α').[2]

The paper is organized as follows. In section 2, we study in some detail the four-dimensional situation. The formulation can be generalized to that involving a p-form in D dimensions and its dual. Our point of view is that different gauge-fixings of the same topological symmetry, which involve more fields, give the expression of the supersymmetric action either in the non-Abelian (Donaldson–Witten) form or in the Abelian (Seiberg–Witten) form, which describe the infrared behavior.

In sec. 3, we concentrate on a Yang–Mills field in the eight-dimensional case, and show how a dual formulation can be obtained. In contrast with the four-dimensional case, the invariants of the eight-dimensional case that can be constructed from the quadratic action should be expressible in terms of the classical ones. However string corrections, which regularize the ultraviolet behavior and enrich the infrared properties, can open the gates to quantum effects. Our results indicate the probable existence of a symplectic geometry description analogous to that of the four-dimensional case. In particular, the gauge-fixing of the topological symmetry in the ghost sector leads to a relation between scalar components (defining the "order parameters") and their duals, which generalizes that of four-dimensional case. However the lack of a precise renormalization group argument (which is replaced here by topological arguments), does not allow to specify what information is sufficient to determine the "holomorphic prepotential", although a better understanding of string corrections could help solving this issue. We also comment on the fact that from the eight-dimensional TQFT point of view, the appearance of an extra monopole hypermultiplet in the low-energy theory of Seiberg–Witten is very natural, as first noticed in [7]. However we don't discuss its coupling to the genuine Yang–Mills TQFT in the four-dimensional section, since the way to do it is obvious and doesn't add new ingredients.

Finally, in section 4, we indicate how our point of view also applies to two-dimensional duality, for a coupled scalar and Yang–Mills supersymmetric theory. Obviously, the dimensional reduction to two dimensions of the eight-dimensional theories in their various formulations might be independently interesting.

2. Four dimensions

2.1 The fields and the ungauged-fixed action

Let us consider a Yang–Mills field A in four dimensions. As explained in [8, 9], it is natural to associate a two-form gauge field B_2 with A. This relies on the unification

[2] We would like to thank W. Lerche, S. Stieberger and N. Warner for pointing this out to us and for important discussions and clarifications on the above questions pointed.

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between gauge fields and ghosts, for the fields as well as for their Batalin–Vilkoviski antifields. Indeed, in four dimensions, the ghost expansion of a two-form contains the antifields A_3^{-1} and c_4^{-2} of the gauge field A and of the Faddeev-Popov ghost c, while that of the Yang–Mills field A contains the antifields B_2^{-1}, B_3^{-2} and B_4^{-3} of the two-form B_2 and of its ghost and ghost of ghost Ψ_1^1 and Φ_0^2.

Our aim is to show that the pairing of A with a two-form B_2 gauge field provides an understanding of duality properties in four dimensions. Actually, to write an lagrangean, we must introduce another pair, A_D and B_{D2}. This doubling of degrees of freedom becomes clearer if A is replaced by a p-form gauge field in an arbitrary dimension D, and if we generalize the idea of ghost unification, as in [9]. We will go back to this in section 3, for the case of a Yang–Mills theory in eight dimensions, and in section 4, which is devoted to the two-dimensional situation.

The one-form A_D cannot be understood as a Yang–Mills field since the curvature G_{A_D} of A_D turns out to be its covariant derivative with respect to A, that is, $G_{A_D} = D_A A_D$, with $D_A = d + [A, .]$, while that of A is the Yang–Mills curvature $F_A = dA + A \wedge A = dA + \frac{1}{2}[A, A]$.

According to [8], the following expansion determines the fields of the theory:

$$\begin{aligned} \tilde{A} &= c + A + B_2^{-1} + \Psi_3^{-2} + \Phi_4^{-3} \\ \tilde{B}_2 &= c_4^{-2} + A_3^{-1} + B_2 + \Psi_1^1 + \Phi_0^2 \end{aligned} \tag{2.1}$$

$$\begin{aligned} \tilde{A}_D &= c_D + A_D + B_{D2}^{-1} + \Psi_{D3}^{-2} + \Phi_{D4}^{-3} \\ \tilde{B}_{D2} &= c_{D4}^{-2} + A_{D3}^{-1} + B_{D2} + \Psi_{D1}^1 + \Phi_{D0}^2 \,. \end{aligned} \tag{2.2}$$

In this expansion, the forms with negative ghost number are antifields. We follow the usual notation that the lower index is the ordinary form degree, which cannot exceed the value of the space dimension, while the upper index is the ghost number. The way we display the ghost expansions in (2.1) and (2.2) clearly indicates the field-antifield pairings, as they were sketched at the beginning of this section.

The symmetry of the theory is defined by a BRST, that is, an s-symmetry operation, where s acts as a differential operator graded by the sum of ordinary form degree and ghost number, and $s^2 = 0$. It turns out that s can often be related to ordinary supersymmetry transformations by the operation of twist.

This s defines classical infinitesimal transformations governed by two one-form parameters ρ and ρ_D, which correspond to the ghosts Ψ_1^1 and Ψ_{D1}^1, and by two zero-form parameters ϵ and ϵ_D, which correspond to the ghosts c and c_D. Here, ϵ is the ordinary Yang–Mills transformation parameter, and c is the associated Faddeev–Popov ghost.

In [8], a topological s-symmetry for the fields in (2.1) and (2.2) has been introduced, by means of the master equation of the field-antifield dependent Batalin–Vilkoviski lagrangean:

$$\mathcal{L} = \mathrm{Tr}\Big(\tilde{B}_{D2} \wedge \tilde{B}_2 + \tilde{B}_2 \wedge D_{\tilde{A}} \tilde{A}_D + \tilde{B}_{D2} \wedge F_{\tilde{A}} \Big)\Big|_4^0 \,. \tag{2.3}$$

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We have

$$s\int \mathcal{L} = 0\,, \qquad s\phi = \frac{\delta \int \mathcal{L}}{\delta \psi}\,, \qquad s\psi = \frac{\delta \int \mathcal{L}}{\delta \phi}\,, \tag{2.4}$$

where, generically, ψ is the antifield of ϕ. Since (2.3) is a lagrangean of the first order, its equations of motion formally determine the BRST equations for all fields and antifields.

A compact way of writing the action of s on all fields and antifields, which leaves the lagrangean (2.3) invariant, is:

$$\begin{aligned} s\tilde{A} &= -d\tilde{A} - \tilde{A}\wedge\tilde{A} + \tilde{B}\,, & s\tilde{B}_2 &= -D_{\tilde{A}}\tilde{B}_2\,, \\ s\tilde{A}_D &= -d\tilde{A}_D - [\tilde{A}, \tilde{A}_D] + \tilde{B}_{D2}\,, & s\tilde{B}_{D2} &= -D_{\tilde{A}}\tilde{B}_{D2} - [\tilde{A}_D, \tilde{B}_2]\,. \end{aligned} \tag{2.5}$$

The s-variation of each one of the fields and antifields is then obtained by a further expansion in ghost number. It give in particular the topological transformations $sA = \Psi^1_1 + \cdots$ and $sA_D = \Psi^1_{D1} + \cdots$

Observing the way $\tilde{A}$ and $\tilde{A}_D$ transform, it is justified to examine if we can add to the lagrangean (2.3) a term $\mathcal{F} = \mathcal{F}(\tilde{B}_2, \tilde{B}_{D2})$. Here, $\mathcal{F}$ is a group scalar. By assumption, we chose it to be metric-independent.

The condition that we have an s-invariant action, such that $s^2 = 0$, with a non-Abelian symmetry, implies that $\mathcal{F}$ is a function of $\tilde{B}_2$ only: $\mathcal{F} = \mathcal{F}(\tilde{B}_2)$. Moreover, $\mathcal{F}$ must fulfil the condition:

$$\left[\tilde{B}_2, \frac{\delta\mathcal{F}(\tilde{B}_2)}{\delta\tilde{B}_2}\right] = 0\,. \tag{2.6}$$

This condition holds in particular when the gauge group is SU(2). Then, the new lagrangean is:

$$\begin{aligned} \mathcal{L} = \mathrm{Tr}\Big(&\tilde{B}_2\wedge\tilde{B}_{D2} \;+ \tilde{B}_2\wedge D_{\tilde{A}}\tilde{A}_D + \tilde{B}_{D2}\wedge F_{\tilde{A}} + \mathcal{F}(\tilde{B}_2) + x\tilde{F}_{\tilde{A}}\wedge\tilde{F}_{\tilde{A}} + \\ &+ 2y\tilde{F}_{\tilde{A}}\wedge D_{\tilde{A}}\tilde{A}_D + z\Big(D_{\tilde{A}}\tilde{A}_D\wedge D_{\tilde{A}}\tilde{A}_D + F_A\wedge[\tilde{A}_D, \tilde{A}_D]\Big)\Big)\Big|^0_4\,. \end{aligned} \tag{2.7}$$

For completeness, we have added to the lagrangean purely topological terms, where x, y and z are complex numbers. They can be adjusted in such a way that, eventually, the θ parameter of the theory is defined modulo 2π. We are not interested in this issue, so we will set $x = y = z = 0$ in the following.

The modified topological s-invariance for the lagrangean (2.7) is:

$$\begin{aligned} s\tilde{A} &= -F_{\tilde{A}} + \tilde{B}_2\,, & s\tilde{B}_2 &= -D_{\tilde{A}}\tilde{B}_2\,, \\ s\tilde{A}_D &= -D_{\tilde{A}}\tilde{A}_D + \tilde{B}_{D2} + \frac{\delta\mathcal{F}(\tilde{B}_2, \tilde{B}_{D2})}{\delta\tilde{B}_2}\,, & s\tilde{B}_{D2} &= -D_{\tilde{A}}\tilde{B}_{D2} + [\tilde{B}_2, \tilde{A}_D]\,. \end{aligned} \tag{2.8}$$

Obviously, the asymmetry between $\tilde{B}_2$ and $\tilde{B}_{D2}$ comes from that between $\tilde{A}$ and $\tilde{A}_D$ in the non-Abelian case. Since $\tilde{A}_D$ rotates under Yang–Mills symmetry and has also its own local gauge symmetry, with ghost c_D, it is not a standard Yang–Mills field.

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Provided (2.6) is verified, there is no further restriction on the $\tilde{B}_2$ dependence of $\mathcal{F}$ to have $s^2 = 0$ on all fields. The usual physical requirement is that, after gauge-fixing, the action contains a Gaussian part that is positive-definite. This implies $\mathrm{Im}\, \partial^2 \tilde{F} > 0$. The choice of a given $\mathcal{F}$, as in [1], is based on dynamical requirements, which go beyond phase-space considerations.

If the symmetry is purely Abelian, there is a formal symmetry between A and A_D, and $\mathcal{F}$ can depend on $\tilde{B}_2$ and $\tilde{B}_{D2}$. However, throughout the paper, we consider that the commuting case is a limiting case of the non-Abelian one, and we will consider that $\mathcal{F}$ is a function of $\tilde{B}_2$ only.

2.2 Non-Abelian case, with $\mathcal{F} = \mathcal{F}(\tilde{B}_2)$

The lagrangean (2.7) can be considered as a rather sophisticated form of a classical topological invariant. For a non-vanishing $\mathcal{F}$, it actually depends on the fermions of the theory, that is on the topological ghosts and on the antifields. Indeed, by Taylor expansion, we have:

$$\mathcal{F}(\tilde{B}_2)\Big|_4^0 = \mathrm{Tr}\Big(\partial\mathcal{F}(\Phi_0^2)c_4^{-2} + \frac{1}{2}\partial^2\mathcal{F}(\Phi_0^2)\Big(B_2 \wedge B + 2\Psi_1^1 \wedge A_3^{-1}\Big) + \\ + \frac{1}{2}\partial^3\mathcal{F}(\Phi_0^2)B_2 \wedge \Psi_1^1 \wedge \Psi_1^1 + \frac{1}{24}\partial^4\mathcal{F}(\Phi_0^2)\Psi_1^1 \wedge \Psi_1^1 \wedge \Psi_1^1 \wedge \Psi_1^1\Big)\,. \quad (2.9)$$

We use the notation $\partial = \frac{\partial}{\partial\Phi_0^2}$. The apparent complexity of this part of the lagrangean (2.7) is quite analogous to that of consistent anomalies, prior to the understanding of descent equations. One sees that when $\partial^2\mathcal{F} \neq 0$, there are interactions between Φ_0^2 , B_2 and Ψ, and, eventually, A. This occurs even in the commuting case.

The cohomology of s, for its part with ghost number 0, is empty. This is almost obvious if one looks at the way A and A_D transform in (2.5). Thus, we expect that the "classical" lagrangean (2.7) is a closed-term, with a relation to a sort of super Chern–Simons-term. We have:

$$(s - d)\mathcal{F}(\tilde{B}_2) = 0\,. \quad (2.10)$$

This can be seen from the way $\tilde{B}_2$ transforms.

Furthermore, in the commuting limit, we have that $\mathcal{F}(\tilde{B}_2)\big|_4^0$ is by itself the sum of d-closed and s-exact terms. Indeed, in this case, we can get easily the above-mentioned Chern–Simons formula. One uses $\tilde{B}_2 = s\tilde{A} - F_{\tilde{A}}$, and $(s - d)^2 = 0$, and:

$$\begin{aligned}\mathcal{F}(\tilde{B}_2) &= \mathcal{F}((s - d)\tilde{A}) \\ &= \sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0)}{n!}((s - d)\tilde{A})^n \\ &= (s - d)\sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0)}{n!}\tilde{A}((s - d)\tilde{A})^{n-1} \\ &= (s - d)\left(\tilde{A}\sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0)}{n!}\tilde{B}_2^{n-1}\right). \end{aligned} \quad (2.11)$$

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Thus we have

$$\mathcal{F}(\tilde{B}_2)\Big|_4^0 = s\Delta_4^{-1} + d(\ldots) \tag{2.12}$$

and (2.11) gives

$$\begin{aligned}\Delta_4^{-1} = \mathrm{Tr}\Big(&\partial\mathcal{F}(\Phi_0^2)\Phi_4^{-3} + \\ &+ \frac{1}{2}\partial^2\mathcal{F}(\Phi_0^2)\Big(\Psi_3^{-2}\wedge\Psi_1^1 + B_2\wedge B_2^{-1} + A\wedge A_3^{-1} + cc_4^{-2}\Big) + \\ &+ \frac{1}{6}\partial^3\mathcal{F}(\Phi_0^2)\Big(\Psi_1^1\wedge\Psi_1^1\wedge B_2^{-1} + 2A\wedge B_2\wedge\Psi_1^1 + cB_2\wedge B_2 + 2c\Psi_1^1\wedge A_3^{-1}\Big) + \\ &+ \frac{1}{24}\partial^4\mathcal{F}(\Phi_0^2)\Big(c\Psi_1^1\wedge\Psi_1^1\wedge\Psi_1^1\wedge\Psi_1^1\Big)\Big)\,.\end{aligned} \tag{2.13}$$

This expression is quite instructive, and shows the role of antifields. Let us indicate by anticipation that, after the self-dual gauge-fixing that provides, by following the standard rules of the Batalin–Vilkoviski formalism, an action that is suitable for path integration, the antifields will become functions of propagating antighosts. Actually, the self-dual gauge determines values for the antifield of the following type:

$$\begin{aligned}\Psi_3^{-2} &= {}^*(D_A\Phi_0^{-2})\\ A_3^{-1} &= {}^*\Big(D_A\wedge^*\chi_2^{-1+} + [\Phi_0^{-2},\Psi_1^1] + \lambda dc\Big)\\ \Phi_4^{-3} &= {}^*[\Phi_0^{-2},\eta_0^{-1}]\\ B_2^{-1} &= c_4^{-2} = 0\,.\end{aligned} \tag{2.14}$$

Here the constant λ is the usual gauge parameter for fixing the longitudinal degrees of freedom of A.

Finally, to directly obtain the action of s on all fields, it is convenient to decompose all fields in (2.7), and to use the interpretation of the antifields as the sources of the BRST transformation of fields as given by (2.4). This gives:

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big(&B_2\wedge B_{D2}\ + B_2\wedge D_AA_D + B_{D2}\wedge F_A + \tau B_2\wedge B_2 + \\ &+ A_3^{-1}\wedge(\Psi_{D1}^1 + \tau\Psi_1^1 + D_Ac_D - [c,A_D]) + A_{D3}^{-1}\wedge(\Psi_1^1 + D_Ac) + \\ &+ c_4^{-2}(\Phi_{D0}^2 - \partial\mathcal{F}(\Phi_0^2) - [c,c_D]) + c_{D4}^{-2}\left(\Phi_0^2 - \frac{1}{2}[c,c]\right) + \\ &+ B_2^{-1}\wedge(D_{\tilde{A}}\Psi_{D1}^1 - [c,B_{D2}]) + B_{D2}^{-1}\wedge(D_{\tilde{A}}\Psi_1^1 - [c,B_2]) + \\ &+ \Psi_3^{-2}\wedge(D_{\tilde{A}}\Phi_{D0}^2 - [c,\Psi_{D1}^1]) + \Psi_{D3}^{-2}\wedge(D_{\tilde{A}}\Phi_0^2 - [c,\Psi_1^1]) - \\ &- \Phi_4^{-3}[c,\Phi_{D0}^2] - \Phi_{D4}^{-3}[c,\Phi_0^2] + \frac{1}{2}\partial^3\mathcal{F}(\Phi_0^2)F_A\wedge\Psi_1^1\wedge\Psi_1^1 + \\ &+ \frac{1}{24}\partial^4\mathcal{F}(\Phi_0^2)\Psi_1^1\wedge\Psi_1^1\wedge\Psi_1^1\wedge\Psi_1^1\Big)\,.\end{aligned} \tag{2.15}$$

We have defined the matrix:

$$\tau = \frac{\delta^2\mathcal{F}(\Phi_0^2)}{\delta\Phi_0^2\delta\Phi_0^2}\,. \tag{2.16}$$

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Up to now nothing much interesting has been done - we just have introduced $\mathcal{F}$ in the action and explored some simple properties of corresponding BRST algebra. One can't find any interesting duality symmetries in non-Ableian theory. Intersting properties arise only when moving to Ableian phase. As it was mentioned in the introduction in physical theory one shall use RG arguments in order to derive low energy effective action-SW Ableian SUSY gauge theory. In topological setup one can try to gauge fix all fields in the Cartan subalgebra but it turnes out that corresponding theory is related to the low energy solution mentioned above only when non-zero $\mathcal{F}$ is intruduced (see footnote below).

2.3 Restriction of fields in the Cartan subalgebra

Let us now consider the possibility of using the topological symmetry to do a prior gauge-fixing, which reduces all fields in the Cartan subalgebra of the gauge group.[3]

Once all fields are commuting, there is a rotational symmetry between A and A_D, when $\mathcal{F} = 0$. We will see that the interpretation of Abelian duality is the freedom in using different gauge-fixings for the topological symmetry of both A and A_D.

In the physical theory, the moduli space of vacua is parametrized by order parameters. These parameters are the expectation values of invariant polynomials of Φ; we denote the eigenvalues of the matrix Φ_0^2 and Φ_{0D}^2 by a and a_D respectively. The topological symmetry allows one to impose a relation $c_D = 0$, as a gauge choice. A simple look at the way c_D transforms under s, (sc_D is given by the term in factor of c_4 in (2.15)), indicates that the BRST invariance implies a relation between the ghosts of ghosts (the scalars in supersymmetry language)

$$a_D = \frac{\partial \mathcal{F}}{\partial a}\,. \tag{2.17}$$

The choice of another combination of c and c_D, which one would gauge-fix to zero instead of c_D, is also possible.

2.4 The topological invariance and the gauge-fixing freedom

We first consider the classical lagrangean $\mathcal{L}_{cl}$, obtained by putting all ghosts and antifields equal to zero in (2.7) and assuming that τ is constant:

$$\mathcal{L}_{cl} = \mathrm{Tr}\left(B_2 \wedge B_{D2} \ + B_2 \wedge D_A A_D + B_{D2} \wedge F_A + \frac{1}{2} B_2 \wedge \tau B_2\right). \tag{2.18}$$

[3]If the bundle is non-trivial, one must be careful in such a "brutal" gauge-fixing. Indeed, it cannot be decomposed as the sum of line bundles, and it is not possible to globally choose the gauge that reduces the theory to the Abelian one. What one can do, instead, is to choose the Abelian gauge on the complement to the finite number of points (point-like instantons); later, one needs to integrate over these points. Interestingly, it seems that for some magic reason, the gauge-fixing in the Cartan subalgebra, together with the introduction of the monopole hypermultiplet in our formalism, when we have a non-vanishing $\mathcal{F}$, is self-consistent, and the above remark can be ignored. We wish to understand this better and will give some hint later from the eight-dimensional perspective.

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This lagrangean, after elimination of B_2 and B_{D2}, is d-closed. It can be called a classical topological term, and it needs a BRST-invariant gauge-fixing.

In the case of interest, that is when τ is not a constant, (2.7) is the sum of a d-closed *and* an s-exact term. The latter cannot be interpreted as part of a gauge-fixing term, since it only involves the fields and antifields of the geometrical part of the BRST symmetry, prior to the introduction of the cohomologically trivial antighost sector. It follows that (2.7) extends in a supersymmetric way the definition of a topological term. We are in a context that is slightly more general than that of the standard Yang–Mills TQFT [6], which relies on the BRST invariant quantization of an ordinary topological term. However, the invariance of the lagrangean (2.7) still corresponds to the invariances under arbitrary shifts of A and A_D, as can be seen from the BRST equations (2.8). Thus as far as the gauge-fixing is concerned, we can extend the strategy as in [6], which can be applied whether τ is field-dependent or not. For a consistent gauge-fixing, one must use the complete action, with its full antifield dependence, which encodes all information about the BRST symmetry.

As already indicated, the novelty is that (2.7) can be gauge-fixed in different ways, which give various supersymmetric formulations that can be related by duality transformations. This property relies on the following characteristics:

- The symmetry on the gauge fields A and A_D is of the topological type, as shown in (2.8); the dependence in the two-forms B_2 and B_{D2} is purely algebraic, so these fields can be eliminated from the action.

- There are several ways to gauge-fix the symmetries on A and A_D, which leave actions of the same type, but with different values of the coupling constants.

All these properties hold at the Abelian as well as at the non-Abelian levels and give different ways of expressing the theory. We can call this property duality *covariance*; however, duality *invariance* only holds if A and A_D are valued in the Cartan subalgebra of the gauge group, a condition that can be realized by using only part of the freedom of the topological symmetry. In the following, we will nevertheless write formula that accommodate the non-Abelian case.

Let us make a technical comment. There are two ways of enforcing gauge conditions: either one replaces the antifields in the lagrangean (2.7) by relevant expressions of the type $\psi = \frac{\delta Z^{-1}}{\delta \phi}$, where the chosen gauge functions determine Z^{-1}, according to the standard Batalin–Vilkoviski construction; or one adds an s-exact term to the lagrangean obtained by setting all antifields equal to zero. The latter way is consistent because, for the type of symmetry that we consider, the antifield dependence is a linear one. Furthermore it is faster, since it avoids the step of introducing the antifields of antighosts. The s-exact term also depends on the chosen gauge functions. Both procedures are actually equivalent and necessitate the definition of a given set of antighosts and of their Lagrange multipliers. The antighost sector, which is BRST

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cohomologically trivial, must be adapted to the gauge choice. As shown in [6], freedom in the possibility of this sector of the theory makes the richness of TQFTs. We will elaborate on this in the next section, since this is precisely the possibility of having different classes of gauge-fixings for different combinations of $\tilde{A}$ and $\tilde{A}_D$ which, will turn out to be the key to obtain mirror-, or duality-, related formulations.

It is worth noting at this point that the Seiberg–Witten holomorphicity properties of the functional $\mathcal{F}$ are understood in the BRST language as follows: general arguments indicate that the BRST cohomology cannot depend on antighosts. Thus it is expected that any dependence of $\mathcal{F}$ on the antighosts Φ_0^{-2} and Φ_{D0}^{-2}, which we shall shortly define, could be absorbed in an irrelevant counterterm. Since Φ_0^{-2} and Φ_{D0}^{-2} can be interpreted as the complex conjugates of Φ_0^2 and Φ_{D0}^2 in the language of untwisted supersymmetry, the property that $\mathcal{F}$ can only depend on the latter fields finally gives the holomorphicity property.

2.5 The gauge-fixing process

The gauge-fixing of all components in A and A_D can be done in various ways, since we have as many parameters in the symmetry as there are modes in the gauge field (up to global excitations), see [6]. Let us now define the antighosts and Lagrange multipliers that are needed for this purpose in the A, B_2 sector. The mirror equations for the A_D, B_{D2} sector are obvious to deduce, and we skip writing them out.

As for the gauge-fixing of longitudinal modes in A, we have the Faddeev–Popov antighost $\bar{c}$, and its Lagrange multiplier b, and $s\bar{c} = b$. In the topological sector, we have antighosts and Lagrange multipliers that complete the ghost spectrum as follows:

$$\begin{array}{ccccc} & & B_2 & & \\ & \Psi_1^1 & & \Psi_1^{-1} & \\ \Phi_0^2 & & \Phi_0^0 & & \Phi_0^{-2} \end{array}$$

and

$$\begin{array}{ccc} & H_1^0 & \\ \eta_0^1 & & \eta_0^{-1} \end{array}$$

The way s acts on the antighost sector is BRST-trivial. It is: $s\Psi_1^{-1} = H_1^0 - [c, \Psi_1^{-1}]$, $sH_1^0 = -[\Phi_0^2, \Psi_1^{-1}]$, $s\Phi_0^{-2} = \eta_0^{-1} - [c, \Phi_0^{-2}]$, $s\eta_0^{-1} = -[\Phi_0^2, \Phi_0^{-2}]$, $s\Phi_0^0 = \eta_0^1 - [c, \Phi_0^0]$ and $s\eta_0^1 = -[\Phi_0^2, \Phi_0^0]$.

A detailed treatment would imply the introduction of antifields for the antighosts, (e.g. Ψ_3^0 for Ψ_1^{-1}, and so on), such that terms can be added to (2.15), which determine the BRST equations of the antighost sector from a master equation. It is needless to write here such terms, which are of the type $\mathrm{Tr}\Psi_3^0 \wedge (H_1^0 - [c, \Psi_1^{-1}])$.

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2.6 "Trivial" gauge-fixing

The first possibility of gauge-fixing that uses the topological freedom is the "trivial" one: we gauge-fix to zero components of A and/or A_D, (which we will denote by $A_{(w)}$ for notational simplicity).

This gauge-fixing is obtained by the following s-exact lagrangean, which gives algebraic terms that automatically gauge-fix to zero relevant ghost combinations for the $A_{(w)}$-sector:

$$s\mathrm{Tr}\Big(\Psi^{-1\mu}_{(w)}A_{(w)\mu}+\bar{c}_{(w)}\Phi^{0}_{(w)0}+\Phi_0^{-2}c_{(w)}\Big)=\mathrm{Tr}\Big(\mathrm{H}^{\mu}_{(\mathrm{w})}\mathrm{A}_{(\mathrm{w})\mu}+\mathrm{b}_{(\mathrm{w})}\Phi^{0}_{(\mathrm{w})0}+\eta^{-1}_{(\mathrm{w})0}\mathrm{c}_{(\mathrm{w})}+ \\ +\Psi^{-1}_{(w)\mu}sA_{(w)\mu}+\eta^{1}_{(w)0}\bar{c}_{(w)}+\Phi^{-2}_{(w)0}sc_{(w)}\Big)\,. \tag{2.19}$$

We will use the gauge $A_{(w)}=A_D=0$ and $c_{(w)}=c_D=0$. Since, from (2.15), $sA_D|_{c=0}=\Psi^1_{D1}+\tau\Psi^1_1$, and $sc_D|_{c_D=0}=\Phi^2_{D0}-\partial\mathcal{F}(\Phi^2_0)$, the integration over the antighosts Ψ^{-1}_μ and Φ_0^{-2} in (2.19) gives the standard relations $\Phi^2_{D0}=\partial\mathcal{F}(\Phi^2_0)$ and $\Psi^1_{D1}=-\tau\Psi^1_1$. These equations shows that, because of the trivial gauge-fixing, the BRST symmetry relates the ghosts of dual formulations through Legendre-transform type formula.

The gauge-fixing in the cartan subalgebra of A and A_D that we already discussed in section 2.3 is of the "trivial" type.

2.7 Self-dual gauge-fixing

After having "trivially" eliminated part of the fields, (for instance A_D), we can use the remaining freedom to impose a self-dual equation on the curvature of the remaining ones, that we will denote as A_e. As explained in [6], the self-dual gauge-fixing must be completed by an ordinary gauge choice for the longitudinal modes. Altogether, this gives four conditions on A_e and exhausts all possible gauge freedom of the system.

We must now introduce, in the context of BRST invariance, the gauge functions $F^+_{A_e}=(dA_e+A_eA_e)^+$ and $\partial^\mu A_{e\mu}$. For this purpose, we must define our notation for the self-dual and anti-self-dual parts of a two-form:

$$B_2^\pm=\frac{1}{2}(B_2\pm^*B_2)\,, \tag{2.20}$$

with (in Euclidian $4D$-space)

$$^*B_{\mu\nu}=\frac{\sqrt{g}\epsilon}{2}\epsilon^{\mu\nu\rho\sigma}B_{\rho\sigma}\,,$$

and thus, $B_2^+\wedge B_2^-=0$, and

$$B_2\wedge{}^*B_2=d^4x\,\sqrt{g}\,B_{\mu\nu}B^{\mu\nu}=B_2^+\wedge B_2^+ + B_2^-\wedge B_2^-\,, \\ B_2\wedge B_2=d^4x\,\sqrt{g}\,B_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}B_{\rho\sigma}=B_2^+\wedge B_2^+ - B_2^-\wedge B_2^-\,. \tag{2.21}$$

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The squared gauge function $(dA_e + A_e A_e)^+$ provides a Yang–Mills type lagrangean plus a topological term, since $^{**} = 1$, and

$$\frac{1}{2} F^+_{A_e} \wedge F^+_{A_e} = F_{A_e} \wedge^* F_{A_e} + F_{A_e} \wedge F_{A_e} \,. \tag{2.22}$$

To enforce the self-duality gauge function, we only need three antighosts and three Lagrange multipliers. We thus need some redefinitions of the degrees of freedom in the antighost sector. (This eventually leads to the possibility of twist.) We decompose:

$$\begin{aligned} \Psi_1^{-1} &\to (\chi_2^{-1+}, \Psi^{-1}) \\ H_1^{-1} &\to (H_2^+, H) \,, \end{aligned} \tag{2.23}$$

where χ_2^{-1+} and H_2^+ are self-dual two-forms, and Ψ^{-1} and H are zero-forms. (From now on we skip the subindex e for the sake of notational clarity). Changing variables as in (2.23) amounts to decomposing a spinorial tensor of rank 2 into its trace and its traceless parts.

One can eliminate the useless antighosts Ψ^{-1} and Φ_0^0 by means of the s-exact lagrangean

$$s\mathrm{Tr}(\Psi^{-1}\Phi_0^0) = \mathrm{Tr}\Big(\mathrm{H}\Phi_0^0 + \eta^1 \Psi^{-1}\Big) \,. \tag{2.24}$$

The remaining part of (2.19) is:

$$\begin{array}{ccccc} & & B_2 & & \\ & \Psi_1^1 & & \chi_2^{-1+} & \\ \Phi_0^2 & & & & \Phi_0^{-2} \end{array} \tag{2.25}$$

$$\begin{array}{ccc} & H_2^+ & \\ & & \eta_0^{-1} \end{array} \tag{2.26}$$

As explained in [6], the gauge-fixing also implies that of the longitudinal components of Ψ_1^1. The s-exact term that enforces this condition is:

$$s\mathrm{Tr}\left(\tau(\Phi_0^2)\chi^{-1}_{\mu\nu}\left(F_A^{+\mu\nu} + \frac{1}{2}H^{+\mu\nu}\right) + \Phi_0^{-2}\left(D_A^\mu \Psi_\mu^1 + \left[\Phi_0^2, \eta_0^{-1}\right]\right)\right) . \tag{2.27}$$

Finally, the gauge-fixing part of the longitudinal degrees of freedom in A must be done by using the ordinary Faddeev–Popov ghost $\bar{c}$ and its Lagrange multiplier.

2.8 The elimination of B_2 and B_{D2}

The classical two-forms B_2 and B_{D2} can be eliminated from the lagrangean (2.15) by Gaussian integrations, for any given choice of the function $\mathcal{F}$. If we put all antifields to zero, this gives a term that is a quadratic form in the curvatures of A and A_D, plus terms depending on the fermions Ψ_1^1 and the derivatives of τ. In what follows we do not consider the dependence on the fermions Ψ_1^1, although they are important as

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explained in [4], in order to completely demonstrate the mapping between the dual formulations. We refer the reader to [2, 3, 4] for more complete elements of the proof than the ones displayed below, which include the verification that the fermionic part of the action, obtained in expanding the s-exact terms which enforce the self-duality gauge-fixing, obey the equivalence between the actions.

The logic is as follows. We first perform a gauge-fixing of all fields in the Cartan-subalgebra (the symbol Tr now means the trace in this space). After this, there are two possibilities: either we impose the trivial gauge conditions $A_D = 0$ and we recover the standard topological term for A, by the elimination of the auxiliary fields B_2 and B_{D2}; or we first integrate out A, (this is the point where it is important to have done a prior trivial gauge-fixing of all fields in the Cartan-subalgebra). This implies that $B_{D2} = d\Lambda$, for some one-form Λ, and we can set $\Lambda = 0$ by using the topological gauge freedom on A, which is not used at the level of the equation of motion $dB_{D2} = 0$. Doing so, A and all its ghosts disappear. This procedure gives lagrangeans that are identical up to the change $\tau \to -1/\tau$ and field redefinitions.

Let us be a little bit more precise. Starting from the lagrangean (2.15), with the above-mentioned restrictions, the integration over B_2 gives:

$$\begin{aligned}\mathcal{L} &\sim \mathrm{Tr}\left(-\frac{1}{2}(B_{D2}+dA_D)\wedge\tau^{-1}(B_{D2}+dA_D)+B_{D2}\wedge F_A\right)\\ &= \mathrm{Tr}\left(-\frac{1}{2}B_{D2}\wedge\tau^{-1}B_{D2}+B_{D2}\wedge(F_A-\tau^{-1}dA_D)-\frac{1}{2}dA_D\wedge\tau^{-1}dA_D\right). \quad (2.28)\end{aligned}$$

If we now use the topological freedom to first gauge-fix $A_D = 0$, we obtain by Gaussian integration:

$$\mathcal{L} \sim \mathrm{Tr}\left(-\frac{1}{2}B_{D2}\wedge\tau^{-1}B_{D2}+B_{D2}\wedge F_A\right) \sim \mathrm{Tr}\left(\frac{1}{2}F_A\wedge\tau F_A\right). \quad (2.29)$$

This is the standard topological term which leads us directly to the τ-dependent Yang–Mills TQFT, by self-dual gauge fixing, as in [6].

However, duality emerges because we can use in a different and more refined way the topological freedom on A and A_D, by gauge-fixing B_{D2} to zero after integrating out A. Starting again from (2.15), the A-integration gives, together with summing over its fluxes:

$$\mathcal{L} \sim \mathrm{Tr}\left(B_2\wedge(d\Lambda+dA_D)+\frac{1}{2}B_2\wedge\tau B_2\right), \quad (2.30)$$

where there is a functional integration over Λ. The latter field can be gauge-fixed to zero, using the topological freedom on A which has yet not been used, and which gives $s\Lambda = \Psi_1^1$. Once $\Lambda = 0$, the integration over B_{D2} gives:

$$\mathcal{L} \sim \mathrm{Tr}\left(-\frac{1}{2}dA_D\wedge\tau^{-1}dA_D\right). \quad (2.31)$$

This lagrangean, after self-dual gauge fixing of A_D, determines the dual theory, with the symmetry $\tau \to -1/t$.

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3. Eight dimensions

In this section, we will give the description of the eight-dimensional Yang–Mills TQFT by extending what we have done in the four-dimensional case. This TQFT has been introduced in [7], in a non-Abelian formulation which is analogous to that of the four-dimensional Donaldson–Witten TQFT. Here, we will investigate how to introduce the idea of duality in this theory.

In eight dimensions, the dual of a one-form is a five-form. If one considers the generalization of the four-dimensional case, we have a pair A,B_6, instead of A,B_{D2}, and a pair A_5,B_2, instead of A_D,B_6, with the following ghost expansions for the fields and antifields that are adapted to the eight-dimensional case:

$$\begin{aligned}
\tilde{A} &= c + A + B_2^{-1} + \Psi_3^{-2} + \Phi_4^{-3} + \Phi_5^{-4} + \Phi_6^{-5} + \Phi_7^{-6} + \Phi_8^{-7} \\
\tilde{B}_6 &= c_8^{-2} + A_7^{-1} + B_6 + \Psi_5^1 + \Phi_4^2 + \Phi_3^3 + \Phi_2^4 + \Phi_1^5 + \Phi_0^6 \qquad (3.1) \\
\tilde{A}_5 &= A_0^5 + A_1^4 + A_2^3 + A_3^2 + A_4^1 + A_5 + \Psi_6^{-1} + \Phi_7^{-2} + \Phi_8^{-3} \\
\tilde{B}_2 &= A_8^{-6} + A_7^{-5} + A_6^{-4} + A_5^{-3} + A_4^{-2} + A_3^{-1} + B_2 + B_1^1 + B_0^2 \,. \qquad (3.2)
\end{aligned}$$

The Batalin–Vilkoviski lagrangean which generalizes (2.3) is:

$$\mathcal{L} = \mathrm{Tr}\left(\tilde{B}_2 \wedge \tilde{B}_6 \; + \tilde{B}_2 \wedge D_{\tilde{A}}\tilde{A}_5 + \tilde{B}_6 \wedge F_{\tilde{A}}\right)\Big|_8^0 . \qquad (3.3)$$

As in the four-dimensional case, this lagrangean is invariant under a topological symmetry for the Yang–Mills field A and the five-form gauge field A_5, qiven by eq. (2.4). Its classical part is equivalent to the topological density $\mathrm{Tr}\Big(D_A A_5 \wedge F_A\Big)$. Moreover, it is clear that we can introduce a "prepotential" $\mathcal{F}(\tilde{B}_2)$, add it to the lagrangean and distort the BRST symmetry as we did in four dimensions.

It seems, however, that duality cannot hold in eight dimensions because of the obvious off-shell asymmetry between the Yang–Mills one-form A and the five-form A_5. Actually, this difficulty can be circumvented, because of the special properties of eight-manifolds, which have been already used in [7, 10] to make an eight-dimensional Yang–Mills TQFT. Before introducing a prepotential, we must explain this.

Assuming that the eight-manifold has $Spin(7)$ holonomy, there is a canonical self-dual closed four-form $\Omega_4(x)$ which is covariantly constant. It can be locally written as follows: we choose a local vielbein such that the metric is $\sum e_i \otimes \epsilon_i$, where the e_i's are one-forms and $i = 1, \ldots, 8$; then $\Omega_4 = T^{ijkl} e_i \wedge \epsilon_j \wedge e_k \wedge e_l$. This form is invariant under the rotations of e_i, which build the subgroup $Spin(7)$ of SO(8). The tensor T is self-dual, $T_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} T^{\alpha\beta\gamma\delta}$, and it can be written in terms of the octonionic structure constants that define the G_2-structure of seven-dimensional manifolds. Such manifolds have a unique covariantly constant three-form ϕ_3, with its dual Ω_4, which can be locally written as $\phi_3 = c^{abc} e_a \wedge e_b \wedge e_c$, $a, b, c = 1, \ldots, 7$, where the c^{abc} are octonionic structure constants. ϕ_3 is invariant under the subgroup G_2 of $SO(7)$, and $\Omega_4 = e_8 \wedge \phi - *\phi$.

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The invariant four-form $\Omega_4(x)$ can be used to decompose any given two-form B_2 into two $Spin(7)$-irreducible components $B_2^{\pm}$, according to $28 = 21 \oplus 7$. B_2^+ and B_2^- can be called self-dual and antiself-dual respectively. The generalization of the four-dimensional decomposition (2.20) is:

$$B_2^{\pm} = \frac{1}{2}(B_2 \pm^{\dagger} B_2)\,, \tag{3.4}$$

with (in Euclidian space):

$$^{\dagger}B_{\mu\nu} = \frac{\sqrt{g}}{2}\Omega_{\mu\nu\rho\sigma}B^{\rho\sigma}\,. \tag{3.5}$$

As in the four-dimensional case, $B_2^+ \wedge B_2^- = 0$, and:

$$\begin{aligned} B_2 \wedge {}^*B_2 &= d^8x\ \sqrt{g}\ B_{\mu\nu}B^{\mu\nu} = B_2^+ \wedge B_2^+ + B_2^- \wedge B_2^- \\ B_2 \wedge {}^{\dagger}B_2 &= d^8x\ \sqrt{g}\ B_{\mu\nu}\Omega^{\mu\nu\rho\sigma}B_{\rho\sigma} = B_2^+ \wedge B_2^+ - B_2^- \wedge B_2^-\,. \end{aligned} \tag{3.6}$$

Obviously, the symbols * now denotes the ordinary Hodge duality operation in eight dimension.

Now comes the new features. Using Ω_4, one can decompose the gauge field A_5 according to:

$$A_5 = \Omega_4 \wedge A_D + (A_5 - \Omega \wedge A_D)\,, \tag{3.7}$$

where the one-form gauge field A_D is defined as:

$$A_D = {}^*(\Omega_4 \wedge {}^*A_5)\,. \tag{3.8}$$

The idea is to first use the topological symmetry to gauge-fix to zero $(A_5 - \Omega \wedge A_D)$, that is, to enforce the following gauge-fixing:

$$A_5 = \Omega_4 \wedge A_D\,. \tag{3.9}$$

Actually, a simple counting of commuting and anticommuting degrees of freedom in $\tilde{A}_5$ and $\tilde{B}_6$, together with the properties that the ghosts in $\tilde{B}_6$ are topological ghosts for A_5, indicates that one can indeed perform the following gauge fixing

$$\begin{aligned} \tilde{A}_5 &= \Omega_4 \wedge \tilde{A}_D \\ \tilde{B}_6 &= \Omega_4 \wedge \tilde{B}_{D2}\,. \end{aligned} \tag{3.10}$$

The gauge-fixing of $\tilde{A}_5$ down to a one-form $\tilde{A}_D$, and of $\tilde{B}_6$ down to $\tilde{B}_{D2}$ means that that, some of the ghost contained in $\tilde{A}_5$ and $\tilde{B}_6$ are used, after having introduced all necessary antighosts. In the BRST framework, they disappear from the theory because of the algebraic equations of motion stemming from the BRST invariant way of enforcing (3.10).

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The degrees of freedom which are contained in the expansions of of $\tilde{A}_5$ and $\tilde{B}_6$, and which remain ungauged-fixed after the gauge-fixing of A_5 to A_D, can be effectively reorganized in the following expansions:

$$\tilde{A}_5 = \Omega_4 \wedge \tilde{A}_D = \Omega_4 \wedge (c_D + A_D + B^{-1}_{D2} + \Psi^{-2}_{D3} + \Phi^{-3}_{D4}) \,, \tag{3.11}$$

$$\tilde{B}_6 = \Omega_4 \wedge \tilde{B}_{D2} = \Omega_4 \wedge (c^{-2}_{D4} + A^{-1}_{D3} + B_{D2} + \Psi^{1}_{D1} + \Phi^{2}_{D0}) \,. \tag{3.12}$$

Notice that the classical forms B_6 and B_{D2} have the same number of degrees of freedom, but different gauge symmetries.

This gauge-fixing reduces the lagrangean to:

$$\mathcal{L} = \mathrm{Tr}\Omega_4 \wedge \left(\tilde{B}_2 \wedge \tilde{B}_{D2} \ + \tilde{B}_2 \wedge D_{\tilde{A}} \tilde{A}_D + \tilde{B}_{D2} \wedge F_{\tilde{A}} + \mathcal{F}(\tilde{B}_2) \right) \Big|^0_8 \,. \tag{3.13}$$

This is a multiplication by Ω_4 of a lagrangean which is very similar to that we have studied in four dimensions. We have actually incorporated a term $\Omega_4 \wedge \mathcal{F}(\tilde{B}_2)$. By assuming the proportionality of this term to the four-form Ω_4, we eliminate the possibility that the lagrangean contains topological terms proportional to powers in B_2 higher than two. We will soon discuss the relevance of supressing this condition.

For $\mathcal{F} = 0$, the classical part of $\mathcal{L}$ is equivalent to the topological term $\Omega \wedge \mathrm{Tr} F_A \wedge F_A$, that is, the starting point of the TQFT in [7].

For $\mathcal{F} \neq 0$, the situation is analogous to that in four dimensions, but the trivial gauge-fixings must be combined to self-dual gauge-fixings which are specific to eight dimensions. The latter is realized by an s-exact term of the following type:

$$\left\{ Q, \mathrm{Tr} \left(\tau \chi^{-1+}_{\mu\nu} \left(F^{+\mu\nu}_{A_e} + \frac{1}{2} H^{+\mu\nu} \right) + \Phi^{-2}_0 \left(D^{\mu}_{A_e} \Psi^1_{\mu} + [\Phi^2_0, \eta^{-1}_0] \right) \right) \right\} . \tag{3.14}$$

The symbol $^+$ is now defined as in (3.4). (In fact, as explained in [7], there are 7 independent degrees in freedom $\chi^{-1+}_{\mu\nu}$ and $H^+_{\mu\nu}$, according to the $Spin(7)$-decomposition $\underline{28} = \underline{21} \oplus \underline{7}$.) Using (3.6), one easily sees, by repeating the steps that we detailed in the four-dimensional case, that the gauge-fixed action is $\int \frac{1}{g^2} (F_{A_e} \wedge {}^*F_{A_e} + \ldots)$, where the value of g^2 depends on the chosen combination of A and A_D for A_e. The notation $\ldots$ stands for terms that make the action identical, up to twist, to that of the $D = 8$ SSYM theory, that is the dimensional reduction to eight dimensions of $N = 1$ $D = 10$ SSYM theory.

As for the generalization of (2.17), we have exactly the same relation:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \,. \tag{3.15}$$

This equation is consistent the fact that $\Omega_4 \wedge \mathcal{F}$ is an eight-form. Indeed, within the BRST interpretation, the mean values of the ghosts of ghosts, $a_D = \langle \Phi^2_{D0} \rangle$ and $a = \langle \Phi^2_0 \rangle$ are scalars with ghost number two, that is, two-forms, and thus (3.15) is dimensionally meaningful.

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We thus formally find that duality properties can hold in the eight-dimensional Yang–Mills TQFT. The parallel with the four-dimensional case is of course striking. It is due the possible proportionality of the lagrangean to Ω_4. Abelian duality will be obtained by an initial gauge-fixing of the fields in the Cartan subalgebra of the gauge group.

However, one can go further: we have the possibility of adding to the lagrangean (3.13), which already includes the term $\Omega_4 \wedge \mathcal{F}(\tilde{B}_2)$, an SO(8)-invariant term $\mathcal{G}(\tilde{B}_2)|_8^0$, which is not proportional to Ω_4. This gives new interactions to the theory, but no modification of the quadratic part of the action. However, the lagrangean depend on quartic interactions in B_2, and we now look how these terms can be handled in procedures analogous to those that led us to dual lagrangeans in the four dimensional case. The lagrangean is:

$$\mathcal{L} = \mathrm{Tr}\Big(\Omega_4 \wedge (\tilde{B}_2 \wedge \tilde{B}_{D2} \ + \tilde{B}_2 \wedge D_A \tilde{A}_D + \tilde{B}_{D2} \wedge F_{\tilde{A}} + \mathcal{F}(\tilde{B}_2)\ ) + \mathcal{G}(\tilde{B}_2)\Big)\Big|_8^0 . \quad (3.16)$$

As before, we gauge-fix all fields in the Cartan subalgebra, and we still do not include the fermion terms in our discussion, which are proportional to higher derivatives of $\mathcal{F}$ and G. Then, as a first possibility, we can set $A_D = 0$. After elimination of B_{D2}, this gives a classical lagrangean :

$$\mathcal{L} = \mathrm{Tr}\Big(\Omega_4 \wedge \mathcal{F}(dA) + \mathcal{G}(dA)\Big) . \quad (3.17)$$

The latter can be gauge-fixed in the octonionic self-dual way as in [7].

As a second possibility, we first integrate over A. This gives, by summing over all fluxes, $B_6 = d\Lambda_5 = \Omega_4 \wedge d\Lambda$, and as a last step, we use the gauge freedom on A_D to set $\Lambda = 0$. (The analysis about the way to use all ghosts is as in four dimensions). At this point, the lagrangean is:

$$\mathcal{L} = \mathrm{Tr}\Big(\Omega_4 \wedge (B_2 \wedge dA_D + \mathcal{F}(B_2)) + \mathcal{G}(B_2)\Big) . \quad (3.18)$$

If $\mathcal{G} = 0$, the discussion is exactly as in four dimensions, apart from the overall multiplication by Ω_4. If $\mathcal{G} \neq 0$, we cannot perform exactly the quartic integration. However, one can treat these term as a perturbation, and replace, in a first approximation, the argument of $\mathcal{G}$ by $B_2 = \tau^{-1} dA_D$.

Now comes the physically interesting question. The eight-dimensional theory is non-renormalizable and it is infrared-free, as indicated by power counting. From the other side, the topological theory described above is basically the same as the four-dimensional theory: its lagrangean is the product of the $Spin(7)$-invariant 4-form Ω_4 with a lagrangean very similar to the four-dimensional one. It is tempting to conclude, based on the possibility of the gauge-fixing procedure described above, that the eight-dimensional Abelian topological theory can give an analogue to the Seiberg–Witten "infrared description". It should be kept in mind that our eight-dimensional theory is not really topological from the beginning, since only the metric deformations

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that preserve the $Spin(7)$-structure are allowed. Thus arguments relating the ultraviolet theory to the infrared one seem difficult to use, although an "Abelianization" can be achieved by a gauge-fixing which take into account the delicate questions of the treatment of pointlike instantons. At present we unfortunately lack an explicit derivation of the infrared finite-dimensional integral from the compactification of the instanton moduli space even in four dimensions, see for example [4].[4]

There is a question that we ignored in our four-dimensional discussion, and which concerns the coupling of supersymmetric matter to the Abelian theory. Another interest of the eight-dimensional Yang–Mills TQFT is to enlight this coupling, which uses a commuting spinor, and generalizes the genuine Yang–Mills TQFT by a modification of the four dimensional self-duality gauge condition, which becomes the equality of the self-dual part of the Yang–Mills curvature to the commuting-spinor current [2]. As first noted in [7], the eight-dimensional Yang–Mills TQFT precisely gives, by dimensional reduction, a TQFT using four-dimensional Seiberg–Witten equations as topological gauge functions. The argument is as follows. By dimensional reduction from eight to four dimensions, the eight-dimensional gauge field determines a four-dimensional gauge field, (made from the first four components A_μ, $1 \leq \mu \leq 4$), and a four-dimensional bosonic complex Weyl spinor κ (made from the remaining four components $5 \leq \mu \leq 8$). Then, the seven octonionic gauge functions, which determine the TQFT in eight dimensions, become four-dimensional:

$$F^{a+}_{\mu\nu} = f^{a}_{bc}{}^{+} \kappa^b \gamma_{\mu\nu} \kappa^c \gamma^\mu D_\mu \kappa^a = 0 \,. \tag{3.19}$$

Here the indices μ, ν run from 1 to 4, and a, b, c are Lie algebra indices; f^a_{bc} are the Lie algebra structure constants. These equations are non-Abelian, and differ from those given in [2] in the commuting limit. Thus, it is useful to explain in more detail how it could allow us to obtain the *Abelian* Seiberg–Witten equations, for which A is just a U(1) gauge field, rather than (3.19), where A is non-Abelian. This is indeed quite simple, at least formally. One must do a first gauge-fixing in the eight-dimensional theory which sets equal to zero the components of A^a_μ with $1 \leq \mu \leq 4$ that are not in the Cartan subalgebra, and, for $\mu, \nu \geq 5$, all combinations of A^a_μ, say, $\mathcal{A}^a_\mu = \sum_b c^a_b A^b_\mu$, that are such that $f^a_{bc} \mathcal{A}^b_\mu \mathcal{A}^c$ is also not valued in the Cartan subalgebra. This algebraic, or "trivial", gauge-fixing in the Cartan subalgebra also eliminates the corresponding ghosts in the TQFT. Then, for the remaining fields, one uses the octonionic gauge function. This automatically gives Abelian Seiberg–Witten equations.

[4]Note that, although the ultraviolet TQFT is not Lorentz-invariant in eight dimensions, since it explicitly depends on Ω_4, its untwisting is possible and one recovers the ordinary supersymmetric theory that is perfectly Lorentz-invariant as was shown in [7]. If our eight-dimensional is a product of two four-manifolds, or if it is a fibration over a four-dimensional base, we can think about this theory as a four-dimensional one, such that the integral $\int_{fibre} \Omega_4$ gives the coupling constant in four dimensions. In this way, we can recover the four-dimensional topological gauge theory with all its properties.

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It is best to give an example. Let us consider the SU(2) case. In the non-Abelian phase, both A and κ are an SU(2)-triplet. The gauge-fixing sets $A^{(1)} = A^{(2)} = 0$ and $\kappa^{(3)} = \kappa^{(1)} - \kappa^{(2)} = 0$. If we define $\kappa = \kappa^{(1)} = \kappa^{(2)}$, then the remaining topological gauge freedom on $A^{(3)}$ and κ can be used, and the octonionic gauge function gives:

$$F^{+}_{\mu\nu} = {}^{+}\kappa\gamma^{\mu\nu}\kappa\gamma^{\mu}D_{\mu}\kappa = 0\,, \tag{3.20}$$

where $F = dA^{(3)}$ as a result of the first gauge-fixing on A. This shows that the non-Abelian eight-dimensional Yang–Mills theory formally gives the four-dimensional TQFT with "Abelian" Seiberg–Witten gauge-functions, provided a preliminary gauge-fixing has been done to restrict all equations in the Cartan subalgebra of the Yang–Mills group.

Let us return to the eight-dimensional case and see how the theory may depend on higher derivative couplings. As an example, in the gauge-fixed theory, we may need a non-topological quartic term:

$$t^{\mu_1\mu_2\mu_3\mu_4\mu_4\mu_5\mu_6\mu_7\mu_8}(\Phi_0^2)F_{\mu_1\mu_2}F_{\mu_3\mu_4}F_{\mu_5\mu_6}F_{\mu_7\mu_8}\,. \tag{3.21}$$

Such terms were computed in [14] for string compactifications. The appearance of the tensor t is actually well known in string theory [13]. It is made of the product of a certain trace of γ matrices, and it is related to the four-form Ω_4.

One can explain the relevance of terms as in (3.21) as follows: if one starts from the eight-dimensional action TQFT, which can be twisted in an action with eight-dimensional supersymmetry, one soon realizes that non-topological quartic counterterm such that (3.21) are needed, for instance by one-loop corrections.

From our view-point, one should be able to obtain such terms by adding an s-exact counterterm, which was absent at the tree-level[5]. The way to incorporate such terms is quite simple: we can change the definition of $F^{\mu\nu}_{A_e}$ in the part (3.14) of the lagrangean, which enforces the self-duality gauge condition, as follows:

$$F_{A_e\mu\nu} = \partial_{[\mu}A_{e\nu]} \to F_{A_e r\mu\nu} = F_{A_e\mu\nu} + t_{\mu\nu\alpha\beta\gamma\delta\rho\sigma}(\Phi_0^2)F^{\alpha\beta}_{A_e}F^{\gamma\delta}_{A_e}F^{\rho\sigma}_{A_e}\,. \tag{3.22}$$

Indeed, squaring the self-dual part $F^{+\mu\nu}_{A_e r}$ produces, in addition to the standard Yang–Mills lagrangean, the term (3.21). Additional fermionic terms are generated: they are very easy to find by expansion of the new s-exact term. There are of course terms of degree F^6 that are produced. They are of order α'^2, thus corresponding string computations are required. Actually, it is a very interesting question to find out whether all string corrections that are needed to renormalize the theory and that can be organized as a formal series in α' with higher powers of F, can be obtained

[5] An example of the necessity of improving s-exact terms is when one renormalizes the Yang–Mills theory in a non linear gauge: higher-order ghost interactions must be introduced by mean of a BRST-exact term, in order to compensate for the divergences in the four ghost vertex occurring in such gauges.

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by a replacement of F_A in the self-duality condition, which is analogous to (3.22). (It is possible that the Born–Infeld type lagrangean can emerge in such a process.) Here, we simply note that the duality considerations of this section suggest that, when both quadratic and quartic couplings are introduced at the leading order, one has the relation: $\tau_D = -1/\tau$, $t_{D8} \sim t_8/\tau^4$. It would be interesting to verify if the dual formulation that we have introduced in this section fits in the context of string theory, together with these transformation rules (which unfortunately are not exact since quartic interactions do not allow us to exactly integrate out the auxiliary fields).

Finally, we recognize that having the four-form Ω_4 is an essential building block of the theory. It could be the eight-dimensional projection of a self-dual propagating four-form of a ten-dimensional theory. More precisely, we may think of a ten-dimensional theory, whose action is of a Chern–Simons type, function of Ω_4, and of two two-forms gauge fields C_2 and C_2', and has the following expression:

$$\int_{10} d\Omega_4 \wedge C_2 \wedge dC_2' \,. \tag{3.23}$$

This introduces the interesting case of theories that involve a five-form curvature $d\Omega_4 + C_2 \wedge dC_2' - C_2' \wedge dC_2$, with a non trivial dependence on consistent anomalies.

To summarize this section, we have seen that the topological freedom for the large set of fields A, A_5, B_2 and B_6 allows a duality picture in eight dimensions, with a clear relationship with the case of four dimensions.

4. Two dimensions

We can dimensionally reduce to two dimensions the theories that we have introduced in four and eight dimensions. However, we can also consider our duality machinery directly in two dimensions.

We thus start with a two-dimensional gauge field A. Its ghost expansion is $\tilde{A} = c + A + \varphi_2^{-1}$. Its dual is, formally, a form of degree -1, $\tilde{Z}_{-1}$, with no classical content. Indeed, $\tilde{Z}_{-1}$ can only contains fields with negative ghost numbers, $\tilde{Z}_{-1} = \Phi_2^{-3} + \Psi_1^{-2} + W_0^{-1}$. These fields can be identified as the antifields of a two-form W_2, of its ghost Ψ_1^1, and its ghost of ghost Φ_0^2. In turn, the existence of W_2 implies that of its dual, which is a scalar that we call φ. The existence of φ could have been directly inferred from that of φ_2^{-1} in $\tilde{A}$. So, introducing a two-dimensional gauge field A leads us to defining:

$$\begin{aligned} \tilde{A} &= c + A + \varphi_2^{-1} \\ \tilde{\varphi} &= c_2^{-2} + A_1^{-1} + \varphi \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \tilde{Z}_{-1} &= \Phi_2^{-3} + \Psi_1^{-2} + W_0^{-1} \\ \tilde{W}_2 &= \Phi_0^2 + \Psi_1^1 + W_2 \,. \end{aligned} \tag{4.2}$$

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Now we also want (string coordinates) scalars that we call X; we introduce their duals, which we call Y (we could have denoted $Y = X_D$). The additional one-forms, which play a role analogous to that of the two-forms B_2 and B_{D2} in four dimensions and contain in their ghost expansion the antifields of X and Y, are called U_1 and V_1. So, in the string coordinate sector, we can define two sets of "dual" coordinates:

$$\begin{aligned} \tilde{X} &= X + U_1^{-1} + U_2^{-2}\,, \qquad & \tilde{U}_1 &= X_2^{-1} + U_1 + U_0^1 \\ \tilde{Y} &= Y + V_1^{-1} + V_2^{-2}\,, \qquad & \tilde{V}_1 &= Y_2^{-1} + V_1 + V_0^1\,. \end{aligned} \tag{4.3}$$

We can now introduce a TQFT lagrangean density as the following two-form with ghost number zero:

$$\begin{aligned} \mathcal{L}_2 = \mathrm{Tr}\Big(&\tilde{U}_1 \wedge \tilde{V}_1 + \tilde{U}_1 \wedge D_{\tilde{A}}\tilde{X} + \tilde{V}_1 \wedge D_{\tilde{A}}\tilde{Y} + \\ &+ \tilde{\varphi} \wedge \tilde{W}_2 + \tilde{\varphi} \wedge F_{\tilde{A}} + \tilde{W}_2 \wedge (D_{\tilde{A}}\tilde{Z}_{-1} + [\tilde{X}, \tilde{Y}])\Big)\Big|_2^0\,. \end{aligned} \tag{4.4}$$

This lagrangean is of first order and metric-independent; its purely classical part $\mathcal{L}_{cl,2}$ is:

$$\mathcal{L}_{cl,2} = \mathrm{Tr}\Big(U_1 \wedge V_1 + U_1 \wedge D_A X + V_1 \wedge D_A Y + \varphi(W_2 + F_A) + W_2[X, Y]\Big)\,. \tag{4.5}$$

If we eliminate X_1, Y_1 and φ by their algebraic equations of motion $Y = D_A X$, $X = D_A Y$ and $W = F_A$, we obtain:

$$\mathcal{L}_{cl,2} \sim \mathrm{Tr}\Big(D_A X \wedge D_A Y + F_A \wedge [X, Y]\Big)\,. \tag{4.6}$$

We see that (4.6) is closed, contrary to (4.4). The integral over the two-dimensional space of the density (4.6) can be considered as a topological term. This indicates that we are dealing with the lagrangean of a TQFT, and we can repeat the same manipulations as led us to duality in four dimensions.

The invariant Batalin–Vilkoviski-type action is:

$$\mathcal{I}_2 = \int_2 \mathcal{L}_2\,. \tag{4.7}$$

which satisfies the master equation (2.4).

Thus (2.7) is invariant under the topological BRST symmetry:

$$s\tilde{A} = -F_{\tilde{A}} + \tilde{W}_2$$

$$s\tilde{W}_2 = -D_{\tilde{A}}\tilde{W}_2 \tag{4.8}$$

$$s\tilde{X} = -D_{\tilde{A}}\tilde{X} + \tilde{V}_1$$

$$s\tilde{V}_1 = -D_{\tilde{A}}\tilde{V}_1 \tag{4.9}$$

$$s\tilde{Y} = -D_{\tilde{A}}\tilde{Y} + \tilde{U}_1$$

$$s\tilde{U}_1 = -D_{\tilde{A}}\tilde{U}_1 \tag{4.10}$$

$$s\tilde{Z}_{-1} = -D_{\tilde{A}}\tilde{Z}_{-1} + [\tilde{X}, \tilde{Y}] + \tilde{\varphi}$$

$$s\tilde{\varphi} = -D_{\tilde{A}}\tilde{\varphi} + [\tilde{U}_1, \tilde{X}] + [\tilde{V}_1, \tilde{Y}] + [\tilde{W}_2, \tilde{Z}_{-1}]\,. \tag{4.11}$$

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The situation is very reminiscent of that we found earlier, but we have here a Yang–Mills sector and a matter sector. The latter can be understood as the dimensional reduction to two dimensions of, for instance, the eight-dimensional Yang–Mills TQFT, provided one adjusts the number of scalars X and Y.

The gauge symmetry is topological for all fields A, X and Y, and is made of arbitrary shifts defined modulo gauge transformations. The way φ transforms is interesting: it undergoes ordinary gauge rotations, and also rotates proportionally to the parameters of the shift symmetry for X and Y, according to:

$$s\varphi = -[c, \varphi] + [U_0^1, X] + [V_0^1, Y]\,. \tag{4.12}$$

This forbids adding to the lagrangean a φ dependent potential, except in the commuting limit, with a complex φ, in which case we may have a Higgs potential.

By applying the Batalin–Vilkoviski procedure, the antifield-dependent terms allow us to determine a fully gauge-fixed lagrangean. One can chose a gauge that eliminates half of the string coordinates, while, for the remaining ones, one chooses self-dual type ones, $D_A X = *(D_A Y) + \cdots$, i.e. modified holomorphicity conditions. Here, the ... stand for commutators, which may come from dimensional reduction from eight to two dimensions.

We now investigate the question of having more refined gauge functions, for obtaining theories depending on X or Y, with duality transformations between both formulations.

We introduce arbitrary functions $\mathcal{F}(\tilde{W}_2, \tilde{U}_1, \tilde{V}_1)$, and we generalize the lagrangean (2.7) into:

$$\begin{aligned}\mathcal{L}_2 = \mathrm{Tr}\Big(&\tilde{\mathrm{U}}_1 \wedge \tilde{\mathrm{V}}_1 + \tilde{\mathrm{U}}_1 \wedge \mathrm{D}_{\tilde{\mathrm{A}}}\tilde{\mathrm{X}} + \tilde{\mathrm{V}}_1 \wedge \mathrm{D}_{\tilde{\mathrm{A}}}\tilde{\mathrm{Y}} + \tilde{\varphi} \wedge \tilde{\mathrm{W}}_2 + \\ &+ \tilde{\varphi} \wedge F_{\tilde{A}} + \tilde{W}_2 \wedge (D_{\tilde{A}}\tilde{Z}_{-1} + [\tilde{X}, \tilde{Y}]) + \mathcal{F}(\tilde{W}_2, \tilde{U}_1, \tilde{V}_1)\Big)\Big|_2^0\,.\end{aligned} \tag{4.13}$$

If we write that $\mathcal{F} = \tau \tilde{W}_2 + \cdots$, we see that the remaining term at the classical level, after the integration over W_2, is:

$$\mathrm{Tr}(\tau \mathrm{F_A})\,. \tag{4.14}$$

This is one way of understanding that we are considering a topological Yang–Mills theory in two dimensions. We will shortly see this from another point of view.

In the scalar sector, we can gauge-fix the fields in various ways, and this provides different formulations. The strength of the coupling of the resulting sigma models is determined by the derivatives of $\mathcal{F}$ with respect to $\tilde{U}_1$ and $\tilde{V}_1$.

Let us be slightly more specific. Consider the following part of the classical lagrangean:

$$\mathrm{Tr}(\varphi \wedge \mathrm{W_2} + \varphi \wedge \mathrm{F_A} + \mathrm{W_2} \wedge [\mathrm{X}, \mathrm{Y}])\,. \tag{4.15}$$

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The classical two-form $W_2 = dz \wedge d\bar{z} W_{z\bar{z}}$ determines its dual, the scalar $W = \epsilon^{z\bar{z}} W_{z\bar{z}}$.

The first way of fixing the gauge, using the topological symmetry, is to set $W_{z\bar{z}} = 0$. This gives the ordinary topological gauge-invariant lagrangean:

$$\mathrm{Tr}(\varphi \wedge \mathrm{F_A})\,. \tag{4.16}$$

Of course the other gauge symmetry in A must be used to gauge-fix this well-known two-dimensional lagrangean, that one sometimes uses to define the topological two-dimensional Yang–Mills theory. Then, the equations of motion imply that φ must be constant, which gives back (4.16).

We could have first eliminated φ and W by their equations of motion. Then, using the topological symmetry on A, with gauge functions $F_{z\bar{z}}$ and $\partial \cdot A$, we would have again directly obtained the two-dimensional Yang–Mills TQFT, with its BRST-invariant gauge-fixing.

The other way of expressing the theory, in a dual formulation, is to gauge-fix A as follows:

$$A_z = g^{-1}\partial_z g\,, \qquad A_{\bar{z}} = h^{-1}\partial_{\bar{z}} h\,. \tag{4.17}$$

The gauge group elements g and h are defined from the Lie algebra-valued scalars W and φ as:

$$g = \exp(\varphi + W)\,, \qquad h = \exp(\varphi - W)\,. \tag{4.18}$$

This gauge is obtained adding to the lagrangean the BRST-exact term $s(\bar{\Psi}_{\bar{z}}(A_z - g^{-1}\partial_z g) + \bar{\Psi}_z(A_{\bar{z}} - h^{-1}\partial_{\bar{z}} h))$. The bosonic part of the lagrangean has a simple expression if one restricts W and φ in the Cartan Lie algebra:

$$\mathrm{Tr}\Big(\varphi \mathrm{W}_{\mathrm{z\bar{z}}} + \varphi \Delta \mathrm{W}_{\mathrm{z\bar{z}}} + \mathrm{W}_{\mathrm{z\bar{z}}}[\mathrm{X}, \mathrm{Y}]\Big)\,. \tag{4.19}$$

The BRST invariance implies the following ghost terms:

$$\mathrm{Tr}\Big(\bar{\Psi}_z(\Psi_z - s(g^{-1}\partial_z g)) + \bar{\Psi}_{\bar{z}}(\Psi_{\bar{z}} - s(h^{-1}\partial_{\bar{z}} h))\Big)\,. \tag{4.20}$$

Another gauge-fixing term for the topological ghost is necessary:

$$s\mathrm{Tr}\Big(\Phi_0^{-2}(D_z\Psi_{\bar{z}} + D_{\bar{z}}\Psi_z)\Big)\,. \tag{4.21}$$

We have a massive fermionic lagrangean, which is the supersymmetric counterpart of (4.19). The term in (4.21) ensures the inversibility of the fermionic propagator, as well as the propagation of the ghost of ghost Φ_0^2 and its antighost Φ_0^{-2}. Moreover a term $s(\bar{c}\Phi_0^0)$ must be added to algebraically eliminate the n unnecessary antighost sector. Of course, terms of the form $\mathcal{F}(\tilde{W})_2^0$ can be added. Then, non-trivial interactions occurs between W and the field Φ_0^2. We also have the option of having a Higgs potential, by having the choice $\mathcal{F}(\tilde{W}, \varphi)_2^0$ in the commuting limit, with a rotational symmetry between W and φ.

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We actually see that the two-form W and the zero-form φ, which we introduced from the general considerations as in [8], have a natural interpretation: they can be used to express the decomposition of the gauge field in longitudinal and transverse parts, which are specific to two dimensions, and the duality transformation exchanges W and φ. In the gauge $W = 0$, φ has also the interpretation of the Lagrange multiplier of the vanishing curvature condition, which is usually interpreted as the characteristic of a Yang–MIlls TQFT in two dimensions.

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